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NOTE

Comments on the Fractional Step Method

The LU decomposition approach of Ref. [1] for the incompressible Navier–Stokes equations leading to the fractional step method is interesting. However, two issues discussed in Ref. [1] need to be reexamined. These are the pressure boundary conditions and the fractional step method time- and spatial-accuracies.

The semi-implicit discrete form of the incompressible Navier–Stokes equations can be written as [1]

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{bmatrix} v^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}, \tag{1}$$

where (A, G, D) are submatrices and (v^{n+1}, p^{n+1}) are the unknown discrete velocity and pressure vectors, respectively.

Perot [1] considered two approximations for Eq. (1) in order to factor it into LU decomposition. The first approximation is

$$\begin{bmatrix} A & (\Delta t A)G \\ D & 0 \end{bmatrix} \begin{bmatrix} v^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \tag{2}$$

and the second approximation, which is referred to as the "generalized block LU decomposition," is

$$\begin{bmatrix} A & (AB)G \\ D & 0 \end{bmatrix} \begin{bmatrix} v^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}. \tag{3}$$

Perot [1] explains that Eq. (2) is a first-order temporal approximation for Eq. (1) with an error term

$$\frac{\Delta t}{2\mathrm{Re}} LGp^{n+1}. \tag{4}$$

Using Eq. (2), Perot arrives at the Neumann boundary conditions for the pressure field

$$\frac{\partial p}{\partial n} = 0. \tag{5}$$

We show that Eq. (5) is not correct boundary condition for the pressure and is a result of an approximation made by the first-order time-accurate fractional step method and Perot [1]. We further show that this approximation is not acceptable in general, but it may be acceptable for high Reynolds number flows.

Following the LU decomposition procedure of Ref. [1], Eq. (1) can be factored directly without approximations as

$$\begin{bmatrix} A & 0 \\ D & -DA^{-1}G \end{bmatrix} \begin{bmatrix} I & A^{-1}G \\ 0 & I \end{bmatrix} \begin{bmatrix} v^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}. \tag{6}$$

Equation (6) approximates Eq. (1) by

$$\begin{bmatrix} A & AA^{-1}G \\ D & 0 \end{bmatrix} \begin{bmatrix} v^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}. \tag{7}$$

A two-step procedure for solving Eq. (6) gives

$$\begin{bmatrix} A & 0 \\ D & -DA^{-1}G \end{bmatrix} \begin{bmatrix} v^* \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \tag{8a}$$

and

$$\begin{bmatrix} I & A^{-1}G \\ 0 & I \end{bmatrix} \begin{bmatrix} v^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} v^* \\ p^{n+1} \end{bmatrix}. \tag{8b}$$

Further simplifications lead to

$$Av^* = r \tag{9a}$$

$$DA^{-1}Gp^{n+1} = Dv^* \tag{9b}$$

and

$$v^{n+1} + A^{-1}Gp^{n+1} = v^*. \tag{9c}$$

Equation (9b) can be easily solved for the pressure if A^{-1} and G commute

$$A^{-1}G = GA^{-1}. \tag{10}$$

Note that Perot's approximations (2) and (3) satisfy and require Eq. (10), respectively. In this case, Eqs. (9b) and (9c) can be written as

0021-9991/95 $6.00

$$DG(A^{-1}p^{n+1}) = Dv^* \tag{11a}$$

and

$$v^{n+1} + G(A^{-1}p^{n+1}) = v^*. \tag{11b}$$

Using the boundary condition $v^* = v^{n+1}$, Eq. (11b) gives

$$\frac{\partial}{\partial n}(A^{-1}p^{n+1}) = 0. \tag{12}$$

By comparing Eqs. (2) and (7), one concludes that Perot [1] assumed that

$$A^{-1} = \Delta t. \tag{13}$$

At first, Eq. (13) seems to be the acceptable assumption for the first-order time-accurate fractional step method, since AA^{-1} in Eq. (17) is

$$AA^{-1} = I - \frac{\Delta t}{2\mathrm{Re}} L. \tag{14}$$

The error term in Eq. (14) is of the order Δt which was also arrived at by Perot, Eq. (4). However, to understand the hidden problems associated with approximation (13), we consider without loss of generality the steady state solution of Eq. (2). At convergence, the steady state solution of Eq. (2) has spatial error in the pressure term as given by Eq. (4). The error term of the pressure does not go to zero as $\partial v/\partial t$ goes to zero in Eq. (2). In other words, the error term (4) is caused by the approximation (13) and need not be there for steady state solutions. It was, however, considered acceptable by the fractional step method and Perot because it was rationalized in connection with the temporal accuracy of the fractional step method. It is obvious now that the approximation of A^{-1} should not be connected with the temporal accuracy of the fractional step method. The error term should be minimized irrespective of the order of temporal accuracy of the fractional step method. This can be achieved by using the highest order time-accuracy possible for A^{-1}. The choice of

$$A^{-1} = \Delta t\left(I + \frac{\Delta t}{2\mathrm{Re}} L\right) \tag{15}$$

used by Perot [1] for matrix B in Eq. (3) seems to be good approximation for A^{-1} since the next order approximation will involve the operator L^2 which is not simple to handle numerically near the boundaries.

We conclude that the approximation given by Eq. (15) should be used for both the first and second-order fractional step methods. Thus, the Neumann boundary condition equation (12) is applicable for both methods. For high Reynolds number flows, however, the error term equation (4) may be numerically very small and

$$A^{-1}p^{n+1} \approx \Delta t p^{n+1}.$$

In this case, Perot boundary condition (5) may be numerically acceptable.

It is observed from the above analysis that Eq. (8) can be rewritten as

$$\begin{bmatrix} A & 0 \\ D & -DG \end{bmatrix} \begin{bmatrix} v^* \\ \phi \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \tag{16a}$$

$$\begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \begin{bmatrix} v^{n+1} \\ \phi \end{bmatrix} = \begin{bmatrix} v^* \\ \phi \end{bmatrix} \tag{16b}$$

with the boundary condition

$$\frac{\partial \phi}{\partial n} = 0, \tag{16c}$$

where

$$\phi = A^{-1}P^{n+1}. \tag{16d}$$

Equations (16a), (16b), and (16c) can be solved for the velocity field v^{n+1} without knowledge about the pressure field or the matrix A^{-1}. Once the velocity field is computed, then the pressure can be recovered from

$$P^{n+1} = A\phi.$$

Thus, no approximation for the matrix A^{-1} is necessary for the solution of the Navier–Stokes equation (1).

REFERENCES

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